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Pfaffianization of the three-dimensional three-wave equation

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Abstract

A pfaffianized version of the three-dimensional three-wave equation is found using Hirota and Ohta's pfaffianization procedure. In addition, n-lump solutions to the pfaffianized system are presented.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently, there has been a growing interest in finding new coupled systems by means of the pfaffianization procedure, which was first introduced by Hirota and Ohta in 1991 [1, 2]. For example, such a procedure has been successfully applied to the Davey–Stewartson equations [3], the discrete KP equation [4] and the self-dual Yang–Mills equation [5], the two-dimensional Toda lattice [6], etc. In addition, the pfaffianized KP hierarchies have been investigated in [7].

The purpose of this paper is to apply this pfaffianization procedure to a three-dimensional three-wave equation. In section 2, the Grammian solutions of the three-dimensional three-wave equation are reviewed. In section 3, the pfaffianized three-wave equations are presented. In section 4, the bilinear pfaffianized three-wave equations are first converted to a system of nonlinear equations by variable transformations, and then the simplest one-lump solutions of the system are presented. Finally, the conclusion is given in section 5.

2. Grammian solutions of the three-dimensional three-wave equation

This section is devoted to reviewing the Grammian solutions of the three-dimensional three-wave equation (three-wave equation). The three-wave equation is [8, 9]

$$\frac{\partial q_i}{\partial X_i} = \gamma_i q_j^* q_k^*, \qquad \frac{\partial q_i^*}{\partial X_i} = \gamma_i q_j q_k, \tag{1}$$

where i, j, k are cyclic and equal to 1, 2, 3 and $*$ means the complex conjugation. The X_i are the characteristic coordinates, usually defined by

$$\frac{\partial}{\partial X_i} = -\partial_t - v_i \cdot \nabla.$$

The γ_i are coupling constants and are scaled to unity in magnitude, i.e. $\gamma_i = \pm 1$. Different choices of the γ_i correspond to reflections. Following [9], we choose $\gamma_1 = \gamma_2 = \gamma_3 = 1$ and consider this system as stationary in three-dimensional space represented by these characteristic coordinates, i.e. we shall generally be working with the characteristic coordinates although the time t occurs in the system. By the dependent variable transformation

$$q_i = \frac{G_i}{F}, \qquad q_i^* = \frac{G_i^*}{F},$$

where F is understood to be real, we have the bilinear form of the equations [9]

$$D_{X_i} F \cdot G_i + G_j^* G_k^* = 0, \qquad D_{X_i} F \cdot G_i^* + G_j G_k = 0, \tag{2}$$

where i, j, k are cyclic permutations of 1, 2, 3, and D is the usual Hirota derivative defined by [2]

$$D_x a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) a(x) b(x')|_{x'=x}.$$

The n-lump solutions to equations (2) can be written in terms of the Grammian determinants [9] and $F = |\mathbf{F}| = |I + H\Phi|$. Here I is an $(n_1+n_2+n_3) \times (n_1+n_2+n_3)$ identity matrix, H is a Hermitian matrix, and Φ is a block diagonal $(n_1+n_2+n_3) \times (n_1+n_2+n_3)$ matrix

$$\Phi = \begin{pmatrix} \int_{X_1}^{\infty} \phi_i^*(u)\phi_j(u)\,\mathrm{d}u & & 0 \\ & \int_{X_2}^{\infty} \psi_k^*(u)\psi_l(u)\,\mathrm{d}u & \\ 0 & & \int_{X_3}^{\infty} \sigma_m^*(u)\sigma_n(u)\,\mathrm{d}u \end{pmatrix},$$

where $\phi_j = \phi_j(X_1), \psi_k = \psi_k(X_2)$ and $\sigma_m = \sigma_m(X_3)$ are arbitrary functions, $i, j = 1, \ldots, n_1; k, l = 1, \ldots, n_2$, and $m, n = 1, \ldots, n_3$.

Now we introduce a different, but essentially equivalent, form of solutions $F = |\mathbf{F}| = |C + \Phi|$, where C is a Hermitian matrix and Φ is as before. The functions G_i and G_i^* in bilinear equations (2) can be taken to be bordered determinants,

$$G_1 = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ \underline{\psi} & \mathbf{F} \end{vmatrix}, \qquad G_1^* = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ \underline{\sigma} & \mathbf{F} \end{vmatrix},$$

$$G_2 = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ \underline{\sigma} & \mathbf{F} \end{vmatrix}, \qquad G_2^* = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ \underline{\phi} & \mathbf{F} \end{vmatrix},$$

$$G_3 = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ \underline{\phi} & \mathbf{F} \end{vmatrix}, \qquad G_3^* = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ \underline{\psi} & \mathbf{F} \end{vmatrix},$$

where

$$\underline{\phi} = \left(\phi_1, \ldots, \phi_{n_1}; 0, \ldots, 0; 0, \ldots, 0\right)^T, \qquad \underline{\psi} = \left(0, \ldots, 0; \psi_1, \ldots, \psi_{n_2}; 0, \ldots, 0\right)^T,$$
$$\underline{\sigma} = \left(0, \ldots, 0; 0, \ldots, 0; \sigma_1, \ldots, \sigma_{n_3}\right)^T, \qquad \underline{\phi}^\dagger = \left(\phi_1^*, \ldots, \phi_{n_1}^*; 0, \ldots, 0; 0, \ldots, 0\right),$$
$$\underline{\psi}^\dagger = \left(0, \ldots, 0; \psi_1^*, \ldots, \psi_{n_2}^*; 0, \ldots, 0\right), \qquad \underline{\sigma}^\dagger = \left(0, \ldots, 0; 0, \ldots, 0; \sigma_1^*, \ldots, \sigma_{n_3}^*\right).$$

The derivatives of F and G_i also give bordered determinants, for instance,

$$\frac{\partial F}{\partial X_1} = \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ \underline{\phi} & \mathbf{F} \end{vmatrix}, \qquad \frac{\partial G_1}{\partial X_1} = -\begin{vmatrix} 0 & 0 & \underline{\phi}^\dagger \\ 0 & 0 & \underline{\sigma}^\dagger \\ \underline{\phi} & \underline{\psi} & \mathbf{F} \end{vmatrix}.$$

These determinants satisfy the Jacobi identity

$$|F| \begin{vmatrix} 0 & 0 & \underline{\phi}^\dagger \\ 0 & 0 & \underline{\sigma}^\dagger \\ \underline{\phi} & \underline{\psi} & \mathbf{F} \end{vmatrix} = \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ \underline{\phi} & \mathbf{F} \end{vmatrix} \begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ \underline{\psi} & \mathbf{F} \end{vmatrix} - \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ \underline{\psi} & \mathbf{F} \end{vmatrix} \begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ \underline{\phi} & \mathbf{F} \end{vmatrix}.$$

This, in fact, is nothing but the first bilinear equation in (2),

$$D_1 F \cdot G_1 = -G_2^* G_3^*.$$

Similarly we can show all the other bilinear equations in (2) hold.

3. Pfaffianized system of the 3-wave equation

We shall present the pfaffianized system of the 3-wave equation (2). For this purpose, we first review the quadratic identities for pfaffians similar to the Jacobi identity. Here we need the identity in the following form [2, 1]:

$$\begin{aligned} &\mathrm{pf}(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2n)\mathrm{pf}(1, 2, \ldots, 2n) \\ &\quad = \mathrm{pf}(a_1, a_2, 1, 2, \ldots, 2n)\mathrm{pf}(a_3, a_4, 1, 2, \ldots, 2n) \\ &\qquad - \mathrm{pf}(a_1, a_3, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, a_4, 1, 2, \ldots, 2n) \\ &\qquad + \mathrm{pf}(a_1, a_4, 1, 2, \ldots, 2n)\mathrm{pf}(a_2, a_3, 1, 2, \ldots, 2n). \end{aligned} \tag{3}$$

Note that this identity has four terms rather than the three terms in the simplest Jacobi identity for determinants.

Now we begin to pfaffianize the 3-wave equation. In section 2, we show that the 3-wave equation (2) has lump-solutions in the form of Grammian determinants. In order to pfaffianize the 3-wave equation, we replace these Grammian solutions with the pfaffians of the Gramm-type. We consider the following pfaffians of the Gramm type:

$$F = (1, \ldots, n_1, 1', \ldots, n_2', \tilde{1}, \ldots, \tilde{n}_3), \tag{4}$$

$$G_i = -(\alpha_j, \bar{\alpha}_k, 1, \ldots, n_1, 1', \ldots, n_2', \tilde{1}, \ldots, \tilde{n}_3), \tag{5}$$

$$G_i^* = -(\bar{\alpha}_j, \alpha_k, 1, \ldots, n_1, 1', \ldots, n_2', \tilde{1}, \ldots, \tilde{n}_3), \tag{6}$$

where i, j, k are cyclic and equal to 1, 2, 3, $n_1 + n_2 + n_3$ is even, and the entries of the above pfaffians are defined by

$$(i, j) = C_{i,j} + \int_{X_1}^{\infty} (f_i g_j - f_j g_i)\,\mathrm{d}x = C_{i,j} + \Phi_{i,j},$$

$$(i', j') = C_{i',j'} + \int_{X_2}^{\infty} (f_i' g_j' - f_j' g_i')\,\mathrm{d}x = C_{i',j'} + \Psi_{i',j'},$$

$$(\tilde{i}, \tilde{j}) = C_{\tilde{i},\tilde{j}} + \int_{X_3}^{\infty} (\tilde{f}_i \tilde{g}_j - \tilde{f}_j \tilde{g}_i)\,\mathrm{d}x = C_{\tilde{i},\tilde{j}} + \Sigma_{\tilde{i},\tilde{j}},$$

$$(i, j') = C_{i,j'}, \qquad (i, \tilde{j}) = C_{i,\tilde{j}}, \qquad (i', \tilde{j}) = C_{i',\tilde{j}},$$

$$(\alpha_1, i) = f_i, \qquad (\bar{\alpha}_1, i) = g_i, \qquad (\alpha_1, i') = (\alpha_1, \tilde{i}) = 0, \qquad (\bar{\alpha}_1, i') = (\bar{\alpha}_1, \tilde{i}) = 0,$$

$$(\alpha_2, i') = f_i', \qquad (\bar{\alpha}_2, i') = g_i', \qquad (\alpha_2, i) = (\alpha_2, \tilde{i}) = 0, \qquad (\bar{\alpha}_2, i) = (\bar{\alpha}_2, \tilde{i}) = 0,$$

$$(\alpha_3, \tilde{i}) = \tilde{f}_i, \qquad (\bar{\alpha}_3, \tilde{i}) = \tilde{g}_i, \qquad (\alpha_3, i') = (\alpha_3, i) = 0, \qquad (\bar{\alpha}_3, i') = (\bar{\alpha}_3, i) = 0,$$

$$(\alpha_m, \bar{\alpha}_n) = (\alpha_m, \alpha_n) = (\bar{\alpha}_m, \bar{\alpha}_n) = 0, \quad \text{for } m, n = 1, 2, 3,$$

where all the C are constants and $C_{kl} = -C_{kl}$.

Differentiation of an element of the pfaffians generally gives a bordered pfaffian:

$$\frac{\partial}{\partial X_1}(i,j) = -(f_i g_j - f_j g_i) = (\alpha_1, \bar{\alpha}_1, i, j), \qquad \frac{\partial}{\partial X_2}(i,j) = \frac{\partial}{\partial X_3}(i,j) = 0,$$

$$\frac{\partial}{\partial X_2}(i',j') = -(f_i' g_j' - f_j' g_i') = (\alpha_2, \bar{\alpha}_2, i', j'), \qquad \frac{\partial}{\partial X_1}(i',j') = \frac{\partial}{\partial X_3}(i',j') = 0,$$

$$\frac{\partial}{\partial X_3}(\tilde{i},\tilde{j}) = -(\tilde{f}_i \tilde{g}_j - \tilde{f}_j \tilde{g}_i) = (\alpha_3, \bar{\alpha}_3, \tilde{i}, \tilde{j}), \qquad \frac{\partial}{\partial X_1}(\tilde{i},\tilde{j}) = \frac{\partial}{\partial X_2}(\tilde{i},\tilde{j}) = 0.$$

With these expressions, we can show that

$$\frac{\partial F}{\partial X_m} = (\alpha_m, \bar{\alpha}_m, \bullet), \qquad \frac{\partial G_i}{\partial X_i} = -(\alpha_i, \bar{\alpha}_i, \alpha_j, \bar{\alpha}_k, \bullet), \qquad \frac{\partial G_i^*}{\partial X_i} = -(\alpha_i, \bar{\alpha}_i, \bar{\alpha}_j, \alpha_k, \bullet), \tag{7}$$

where i, j, k are cyclic and equal to 1, 2, 3, and we have denoted $\{1, \ldots, n_1, 1', \ldots, n_2', \tilde{1}, \ldots, \tilde{n}_3\}$ by $\{\bullet\}$ for simplicity. Substituting the pfaffian expressions (4)–(7) into the left-hand side of the first bilinear equation in (2), we have

$$G_1 \frac{\partial F}{\partial X_1} - F \frac{\partial G_1}{\partial X_1} + G_2^* G_3^* = -(\alpha_1, \bar{\alpha}_1, \bullet)(\alpha_2, \bar{\alpha}_3, \bullet) + (\bullet)(\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_3, \bullet)$$
$$+ (\alpha_1, \bar{\alpha}_3, \bullet)(\alpha_2, \bar{\alpha}_1, \bullet).$$

With the assistance of pfaffian identity (3), we obtain

$$G_1 \frac{\partial F}{\partial X_1} - F \frac{\partial G_1}{\partial X_1} + G_2^* G_3^* = -(\alpha_1, \alpha_2, \bullet)(\bar{\alpha}_1, \bar{\alpha}_3, \bullet).$$

Choosing $(\alpha_1, \alpha_2, \bullet) = H_3$, $(\bar{\alpha}_1, \bar{\alpha}_3, \bullet) = \tilde{H}_2$, we have

$$D_{X_1} F \cdot G_1 + G_2^* G_3^* = -\tilde{H}_2 H_3. \tag{8}$$

Similarly, choosing $H_1 = (\alpha_2, \alpha_3, \bullet)$, $H_2 = (\alpha_3, \alpha_1, \bullet)$, $\tilde{H}_1 = -(\bar{\alpha}_2, \bar{\alpha}_3, \bullet)$, $\tilde{H}_3 = -(\bar{\alpha}_1, \bar{\alpha}_2, \bullet)$, we can deduce that

$$D_{X_2} F \cdot G_2 + G_3^* G_1^* = -\tilde{H}_3 H_1, \qquad D_{X_3} F \cdot G_3 + G_1^* G_2^* = -\tilde{H}_1 H_2, \tag{9}$$

and

$$D_{X_i} F \cdot G_i^* + G_j G_k = -\tilde{H}_k H_j, \tag{10}$$

$$D_{X_i} F \cdot H_i - H_j G_k^* = G_j H_k, \tag{11}$$

$$D_{X_i} F \cdot \tilde{H}_i - \tilde{H}_j G_k = G_j^* \tilde{H}_k, \tag{12}$$

where i, j, k are cyclic and equal to 1, 2, 3. Therefore, equations (8)–(12) constitute the bilinear form of the pfaffianized 3-wave system.

4. The system of equations in nonlinear form and its lump solutions

To transform the pfaffianized 3-wave system (8)–(12) into nonlinear form, we first relate $\tilde{H}_i$ to the complex conjugation of H_i. For this purpose, we impose certain conditions on the functions $f, g, f', g', \tilde{f}$ and $\tilde{g}$ and on the constants C following a similar procedure in [3],

$$\begin{aligned}
&g_i = f_{n_1-i}^*, && g_i' = f_{n_2-i}^{\prime *}, && \tilde{g}_i = \tilde{f}_{n_3-i}^*, \\
&C_{i,j} = -C_{n_1-i,n_1-j}^*, && C_{i',j'} = -C_{n_2-i',n_2-j'}^*, && C_{\tilde{i},\tilde{j}} = -C_{n_3-\tilde{i},n_3-\tilde{j}}^*, \\
&C_{i,j'} = C_{n_1-i,n_2-j'}^*, && C_{i',\tilde{j}} = C_{n_2-i',n_3-\tilde{j}}^*, && C_{i,\tilde{j}} = C_{n_1-i,n_3-\tilde{j}}^*.
\end{aligned} \tag{13}$$

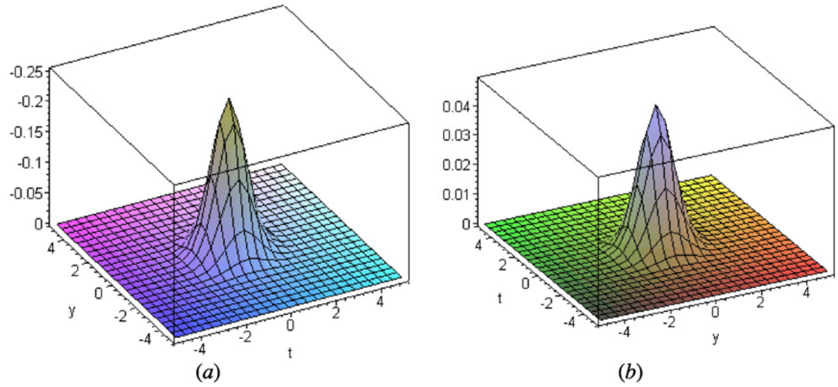

Figure 1. The plots in local profiles: (*a*) the real part of q_1, (*b*) the real part of p_1. $X_1 = 1$, $X_2 = y$, $X_3 = t$.

Under these conditions, we can obtain the identities

$$H_m^* = \tilde{H}_m, \qquad \text{for} \quad m = 1, 2, 3.$$

By the dependent variable transformation

$$q_m = \frac{G_m}{F}, \qquad p_m = \frac{H_m}{F}, \qquad q_m^* = \frac{G_m^*}{F}, \qquad p_m^* = \frac{H_m^*}{F}, \quad m = 1, 2, 3,$$

where F is understood to be real function, we can write the nonlinear equations as

$$\frac{\partial q_i}{\partial X_i} = q_j^* q_k^* + p_j^* p_k, \qquad \frac{\partial q_i^*}{\partial X_i} = q_j q_k + p_j p_k^*, \tag{14}$$

$$-\frac{\partial p_i}{\partial X_i} = p_j q_k^* + q_j p_k^*, \qquad -\frac{\partial p_i^*}{\partial X_i} = p_j^* q_k + q_j^* p_k, \tag{15}$$

where i, j, k are cyclic and equal to 1, 2, 3.

The essential difference between soliton solutions and lump solutions is the absence of (discrete) spectral parameters in the inverse scattering theory. Note that the pfaffianized system (14) and (15) exhibits n-lump solutions in the form of Gramm-type pfaffians in the local profiles if we choose f, f' and $\tilde{f}$ as exponential functions,

$$f_i = c_i\, \mathrm{e}^{-(X_1 - b_i^0)^2}, \qquad f_i' = c_i'\, \mathrm{e}^{-(X_2 - b_i'^0)^2}, \qquad \tilde{f}_i = \tilde{c}_i\, \mathrm{e}^{-(X_3 - \tilde{b}_i^0)^2},$$

where b and c are constants. As an example, consider the simplest case of the lump solutions, i.e., $n_1 = n_2 = n_3 = 2$. In figure 1, we show q_1 and p_1 plotted in the X_2X_3 plane for fixed X_1 with

$$C_{1,2} = C_{1',2'} = C_{\tilde{1},\tilde{2}} = 3/2,$$
$$\begin{pmatrix} C_{1,1'} & C_{1,2'} \\ C_{2,1'} & C_{2,2'} \end{pmatrix} = \begin{pmatrix} C_{1,\tilde{1}} & C_{1,\tilde{2}} \\ C_{2,\tilde{1}} & C_{2,\tilde{2}} \end{pmatrix} = \begin{pmatrix} C_{1',\tilde{1}} & C_{1',\tilde{2}} \\ C_{2',\tilde{1}} & C_{2',\tilde{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$
$$f_1 = \tfrac{1}{2}\,\mathrm{i}\mathrm{e}^{-X_1^2}, \quad \tilde{f}_1 = \tfrac{1}{2}\,\mathrm{i}\mathrm{e}^{-X_2^2}, \quad f_1' = \tfrac{1}{2}\,\mathrm{i}\mathrm{e}^{-X_3^2},$$
$$f_2 = \tfrac{1}{2}\,\mathrm{e}^{-X_1^2}, \quad f_2' = \tfrac{1}{2}\,\mathrm{e}^{-X_3^2}, \quad \tilde{f}_2 = \tfrac{1}{2}\,\mathrm{e}^{-X_2^2},$$

where i denotes $\sqrt{-1}$. A similar behaviour occurs for q_2, q_3, p_2 and p_3.

5. Conclusion

We have presented a pfaffianized system of the three-dimensional three-wave equations. This system contains more fields than the 3-wave equation. In fact, the 3-wave equation can be considered as a reduction of this larger system. This pfaffianized system exhibits n-lump solutions expressed in terms of Gramm-type pfaffians. We have presented the one-lump solution for simplicity. We hope that this pfaffianized 3-wave system (14) and (15) will find physical applications.

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